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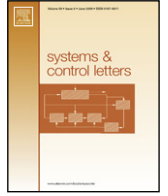
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Canonical realizations by factorization of constant matrices

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ABSTRACT

In van der Schaft and Rapisarda (2011) [3] we showed that a state variable for a LTI system can be computed factorizing a two-variable polynomial matrix computed directly from the system equations. Different factorizations of this matrix yield different state maps, and consequently different state equations; based on this fact we present a unifying point of view on some classical canonical forms for linear systems.

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1. Introduction

Given the set of solutions \mathcal{B} of a system of linear constant-coefficient differential equations in the variables w , a *state map* is a polynomial differential operator $X\left(\frac{d}{dt}\right)$ acting on w which produces a state variable $x := X\left(\frac{d}{dt}\right)w$, for which differential equations of first order in x and zeroth order in w

$$E \frac{d}{dt}x + Fx + Gw = 0, \quad (1)$$

can be written, such that

$$\mathcal{B} = \{w : \mathbb{R} \rightarrow \mathbb{R}^q \mid \text{there exists } x : \mathbb{R} \rightarrow \mathbb{R}^n \text{ such that (1) is satisfied}\}.$$

The notion of *state map* has been introduced and extensively studied in [1]; see also [2]. In [3] the authors used the calculus of bilinear- and quadratic differential forms developed in [4] to show that state maps can be computed by factorizing a constant matrix derived from a two-variable polynomial matrix associated with a special bilinear differential form obtained directly from the system equations. This approach was also applied to Hamiltonian-, adjoint-, and time-reversible systems.

Different factorizations of the two-variable polynomial matrix defined in [3] yield different state maps, and consequently different state equations (1). In this paper we use this fact to present a unifying point of view on some classical canonical forms for linear systems (see for example [5]).

2. Background material

2.1. Trajectory equivalence at 0

We consider systems of differential equations

$$R\left(\frac{d}{dt}\right)w = 0, \quad (2)$$

where $R \in \mathbb{R}^{p \times q}[\xi]$, the ring of $p \times q$ polynomial matrices in the indeterminate ξ . The solution space of (2) is chosen to be $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$, the set of locally integrable trajectories from \mathbb{R} to \mathbb{R}^q ; consequently, $w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ is a (*weak*) *solution* of (2), i.e. $\int_{-\infty}^{\infty} w(t)^T R\left(-\frac{d}{dt}\right)^T \varphi(t) dt = 0$ for all infinitely differentiable test functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}^p$ with compact support. The *behavior* \mathcal{B} associated with (2) is defined by

$$\mathcal{B} := \{w : \mathbb{R} \rightarrow \mathbb{R}^q \mid w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \text{ and (2) is satisfied weakly}\}, \quad (3)$$

and we call (2) a *kernel representation* of \mathcal{B} .

Given $w_1, w_2 \in \mathcal{B}$, the *concatenation* of w_1 and w_2 at time 0, denoted $w_1 \wedge_0 w_2$, is the time-trajectory defined by

$$(w_1 \wedge_0 w_2)(t) := \begin{cases} w_1(t), & t < 0 \\ w_2(t), & t \geq 0 \end{cases}, \quad t \in \mathbb{R}.$$

$w_1, w_2 \in \mathcal{B}$ are *equivalent at time 0*, denoted by $w_1 \sim_0 w_2$, if for all $w \in \mathcal{B}$:

$$w_1 \wedge_0 w \in \mathcal{B} \iff w_2 \wedge_0 w \in \mathcal{B}. \quad (4)$$

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Let $n \in \mathbb{N}$ and $X \in \mathbb{R}^{n \times q}[\xi]$; the polynomial differential operator

$$X \left(\frac{d}{dt} \right) : \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \rightarrow \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$$

$$w \mapsto x := X \left(\frac{d}{dt} \right) w$$

is a *state map* for \mathcal{B} defined by (3) if for all $w_i \in \mathcal{B}$ and corresponding $x_i := X \left(\frac{d}{dt} \right) w_i$, $i = 1, 2$, the following *state property* holds:

$$[x_1(0) = x_2(0)] \quad \text{and} \quad [x_1, x_2 \text{ continuous at } t = 0] \implies [w_1 \sim_0 w_2]. \quad (5)$$

In the definition of state map, $X \left(\frac{d}{dt} \right)$ maps locally integrable trajectories of a behavior to locally integrable trajectories. As shown in the proof of Theorem 2.5 of [3], state maps can be computed integrating by parts $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ -trajectories, and consequently yield absolutely continuous functions. This implies that the smoothness required by the definition can always be attained.

If (5) holds, then x contains all the information necessary to conclude whether any two trajectories in \mathcal{B} admit the same continuation at $t = 0$. For this reason we call $x(0) = (X \left(\frac{d}{dt} \right) w)(0)$ a *state at time 0 corresponding to w* , and we call $\mathcal{X} := \mathbb{R}^n$ the *state space*. If n is minimal among all the state vector dimensions, then the state map is called a *minimal state map*.

2.2. Bilinear differential forms

Let $\Phi \in \mathbb{R}^{p \times q}[\zeta, \eta]$, the set of real two-variable $p \times q$ polynomial matrices in the indeterminates ζ and η ; then $\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell$, where $\Phi_{k,\ell} \in \mathbb{R}^{p \times q}$ and the sum extends over a finite set of indices. The *bilinear differential form (BDF) B_Φ* associated with Φ is defined by

$$B_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$(\varphi, w) \mapsto B_\Phi(\varphi, w) := \sum_{k,\ell} \left(\frac{d^k}{dt^k} \varphi \right)^T \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w.$$

The infinite matrix $\tilde{\Phi}$ whose (k, ℓ) -th block equals $\Phi_{k,\ell}$ is called the *coefficient matrix* of B_Φ . Note that $\tilde{\Phi}$ has only a finite number of nonzero (block-) entries, and that

$$\Phi(\zeta, \eta) = \begin{bmatrix} I_p & I_p \zeta & \dots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \dots \\ \Phi_{1,0} & \Phi_{1,1} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} I_q \\ I_q \eta \\ \vdots \end{bmatrix}.$$

Given a BDF B_Φ , the BDF corresponding to its *derivative*, defined by $B_\Psi(\varphi, w) := \frac{d}{dt} (B_\Phi(\varphi, w))$ is associated, by the product rule of differentiation, to the polynomial matrix $\Psi(\zeta, \eta) := (\zeta + \eta) \Phi(\zeta, \eta)$.

BDFs act on \mathcal{C}^∞ -functions, while the solutions of (2) are in $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$; the mismatch between the degree of differentiability is however not essential in the rest of this paper, where we only use the calculus of two-variable polynomial matrices associated with BDFs.

3. State maps and state equations from factorizations

Given $R \in \mathbb{R}^{p \times q}[\xi]$, we call

$$\Pi(\zeta, \eta) := \frac{R(-\zeta) - R(\eta)}{\zeta + \eta} \quad (6)$$

the *remainder* (see Section 2.2 of [3] for a justification of this terminology). It can be proved that $\Pi(\zeta, \eta)$ is a two-variable polynomial matrix. The fundamental result in [3] is the following.

Theorem 3.1. Let $\mathcal{B} = \ker R \left(\frac{d}{dt} \right)$, with $R \in \mathbb{R}^{p \times q}[\xi]$, and define $\Pi(\zeta, \eta)$ by (6). Let $X \in \mathbb{R}^{n \times q}[\xi]$, $Y \in \mathbb{R}^{n \times p}[\xi]$ be such that

$$\Pi(\zeta, \eta) = Y(\zeta)^T X(\eta); \quad (7)$$

then $X \left(\frac{d}{dt} \right)$ is a state map for \mathcal{B} .

Proof. See Theorem 2.5 of [3]. \square

In order to compute a factorization (7), we proceed as follows. If \tilde{Y}, \tilde{X} are real matrices with n rows, an infinite number of columns, and finitely many entries unequal to zero,

such that $\tilde{\Pi} = \tilde{Y}^T \tilde{X}$, then defining $Y(\xi) := \tilde{Y} \begin{bmatrix} I_p \\ \xi I_p \\ \vdots \end{bmatrix}$ and $X(\xi) =$

$\tilde{X} \begin{bmatrix} I_q \\ \xi I_q \\ \vdots \end{bmatrix}$, also (7) holds. Viceversa, if (7) holds with $Y(\xi) = Y_0 +$

$Y_1 \xi + \dots + Y_L \xi^L$, $X(\xi) = X_0 + X_1 \xi + \dots + X_M \xi^M$, then also $\tilde{\Pi} = \tilde{Y}^T \tilde{X}$ holds, with $\tilde{Y} := \begin{bmatrix} Y_0 & \dots & Y_L & 0_{p \times \infty} \end{bmatrix}$ and $\tilde{X} := \begin{bmatrix} X_0 & \dots & X_M & 0_{n \times \infty} \end{bmatrix}$. Factorizations (7) corresponding to the minimal value $n = \text{rank}(\tilde{\Pi})$ are called *minimal* (or *canonical* as in [4]); note that in this case \tilde{Y} and \tilde{X} are *full row rank*. In general minimality of the factorization of $\tilde{\Pi}$ is necessary but not sufficient for the corresponding state map $X \left(\frac{d}{dt} \right)$ to be minimal. However, if $R(\xi)$ is row-reduced (see Section 6.3.1 of [5]), then any minimal factorization $\Pi(\zeta, \eta) = Y(\zeta)^T X(\eta)$ is such that $X \left(\frac{d}{dt} \right)$ is a minimal state map, see Proposition 2.11 of [3].

Consider a kernel representation (2) where $R(\xi) = R_0 + \dots + R_N \xi^N$, and the remainder (6). It is easy to see that the entries of the remainder only contain powers of ζ and η up to the $(N-1)$ -th one; consequently, we can consider factorizations (7) where $Y(\xi) = Y_0 + \dots + Y_{N-1} \xi^{N-1}$. In Section 2.5 of [3] it has been shown that if $w \in \mathcal{B}$ and $x = X \left(\frac{d}{dt} \right) w$, then a state representation (1) of \mathcal{B} is:

$$\begin{bmatrix} Y_0^T \\ \vdots \\ Y_{N-1}^T \\ 0_{p \times n} \end{bmatrix} \frac{d}{dt} x + \begin{bmatrix} 0_{p \times n} \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} x + \begin{bmatrix} -R_0 \\ R_1 \\ \vdots \\ -(-1)^N R_N \end{bmatrix} w = 0. \quad (8)$$

If the factorization (7) is minimal, then

$$\begin{bmatrix} Y_0^T \\ \vdots \\ Y_{N-1}^T \\ 0_{p \times n} \end{bmatrix} \quad (9)$$

has full column rank, and consequently also a left inverse L partitioned as $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$, with $L_1 \in \mathbb{R}^{n \times Np}$ and $L_2 \in \mathbb{R}^{n \times p}$. Note that (9) has full column rank n if and only if $\begin{bmatrix} Y_0 & \dots & Y_{N-1} \end{bmatrix}^T$ has; consequently, we can assume without loss of generality that L_1 has full row rank. Under these assumptions, premultiplying (8) by

$$\begin{bmatrix} L_1 & L_2 \\ 0_{p \times Np} & I_p \end{bmatrix}, \quad (10)$$

we obtain

$$\frac{d}{dt} x = -L \begin{bmatrix} 0_{p \times n} \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} x - L \begin{bmatrix} -R_0 \\ R_1 \\ \vdots \\ -(-1)^N R_N \end{bmatrix} w$$

$$0 = -Y_{N-1}^T x + (-1)^N R_N w, \quad (11)$$

which also represent the same solution set as (8), since the matrix in (10) can be completed by a suitable $(Np-n) \times (N+1)p$ matrix so as to obtain a unimodular $(N+1)p \times (N+1)p$ matrix (see Section 2.5 of [6]). Eq. (11) is of the form $\frac{d}{dt}x = Fx + Gw$, $0 = Hx + Kw$, often called an *output nulling representation*.

Different factorizations of \tilde{T} and different choices of the left inverse L yield different state maps and consequently different state Eqs. (8), (11); the next section is devoted to articulating the consequences of this elementary fact for the case of SISO systems.

4. Canonical realizations

In the rest of this paper we consider the case $q = 2$, i.e. single-input, single-output systems, and we show that the four classical state space representations, i.e. controller, observer, controllability and observability canonical forms, can be obtained in a straightforward way from special factorizations of the remainder matrix defined by (6). Note that in the case at hand

$$R(\xi) =: [q(\xi) \quad -p(\xi)], \quad (12)$$

where we assume that $\deg(p) = N \geq \deg(q)$. Note also that under these conditions we can take $w_1 = u$ and $w_2 = y$, respectively an input and a (non-anticipating) output variable (see Section 3.3 of [6]). We denote the *Markov parameters* of $\frac{q(\xi)}{p(\xi)}$ by $h_i \in \mathbb{R}$, $i = 0, \dots, i.e.$

$$\frac{q(\xi)}{p(\xi)} =: h_0 + h_1\xi^{-1} + h_2\xi^{-2} + \dots$$

Under these assumptions and conventions, it is a matter of straightforward verification to check that

$$\Pi(\zeta, \eta) = \frac{[q(-\zeta) - q(\eta) \quad -p(-\zeta) + p(\eta)]}{\zeta + \eta},$$

and that the equation given in Box 1 holds.

Moreover, in the factorization (7), $X \in \mathbb{R}^{n \times 2}[\xi]$ and $Y \in \mathbb{R}^{n \times 1}[\xi]$.

We first examine the observability canonical form, see Section 2.2.1 of [5].

Proposition 4.1. *Let $q = 2$, and R be defined by (12). Define \tilde{Y} and \tilde{X} by*

$$\tilde{Y} := \begin{bmatrix} -p_1 & -p_2 & \dots & -p_{N-1} & -p_N & 0 & \dots \\ p_2 & p_3 & \dots & p_N & 0 & \dots & \dots \\ \dots & \dots & \ddots & \ddots & \vdots & \dots & \dots \\ (-1)^{N-1}p_{N-1} & (-1)^{N-1}p_N & 0 & \dots & \dots & \dots & \dots \\ (-1)^N p_N & 0 & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

and

$$\tilde{X} := \begin{bmatrix} h_0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ h_1 & 0 & h_0 & -1 & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & h_1 & 0 & \dots & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & 0 & 0 & 0 & \dots \\ h_{N-1} & 0 & h_{N-2} & 0 & \dots & h_0 & -1 & 0 & \dots \end{bmatrix}.$$

Then $\tilde{T} = \tilde{Y}^T \tilde{X}$. Moreover, there exists a left inverse L of the matrix (9) such that

$$-L \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ -\frac{p_0}{p_N} & \dots & \dots & -\frac{p_{N-1}}{p_N} \end{bmatrix}$$

$$-L \begin{bmatrix} R_0 \\ -R_1 \\ \vdots \\ (-1)^N R_N \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ \vdots & \vdots \\ h_{N-1} & 0 \\ h_N & 0 \end{bmatrix}$$

$$-Y_{N-1}^T = [-p_N \quad 0_{1 \times (N-1)}] \quad (13)$$

$$(-1)^N R_N = (-1)^N [q_N \quad -p_N],$$

i.e. the Eq. (11) are in canonical observability form.

Proof. From the definition of Markov parameters, deduce that $q_i = \sum_{j=0}^{N-i} p_{i+j}h_j$, $i = 0, 1, \dots, N$; from this the first part of the claim follows in a straightforward way.

In order to prove the second part of the claim, define T to be the submatrix consisting of the first N columns of \tilde{Y}^T :

$$T := \begin{bmatrix} -p_1 & -p_2 & \dots & -p_{N-1} & -p_N \\ p_2 & p_3 & \dots & p_N & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ (-1)^{N-1}p_{N-1} & (-1)^{N-1}p_N & 0 & \dots & 0 \\ (-1)^N p_N & 0 & \dots & \dots & 0 \end{bmatrix}.$$

Since $p_N \neq 0$, T is invertible. Denote the j -th element of the canonical basis of \mathbb{R}^N with e_j , $j = 1, \dots, N$; from the triangular structure of T it follows that the first column of T^{-1} , denoted $T_{:,1}^{-1}$ in the following, equals $-e_N \frac{1}{p_N}$. Moreover, from $T^{-1}T = I_N$ it follows that for $k = 0, \dots, N-2$

$$T^{-1} \begin{bmatrix} 0 \\ -p_{N-k} \\ \vdots \\ (-1)^{k-1}p_N \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_{N-k-1} + T_{:,1}^{-1} p_{N-k-1}. \quad (14)$$

Now define $L := [T^{-1} \quad v]$, where $v \in \mathbb{R}^{N \times 1}$ is defined by

$$v := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{p_0}{p_N} \end{bmatrix} (-1)^N \frac{1}{p_N} - (-1)^N \frac{1}{p_N} T^{-1} \begin{bmatrix} 0 \\ -p_1 \\ \vdots \\ (-1)^{N-1}p_{N-1} \end{bmatrix}.$$

It is a matter of straightforward verification using (14) to check that

$$-L \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} \text{ has the required companion form. Moreover, since}$$

$$\begin{bmatrix} -R_0 \\ R_1 \\ \vdots \\ -(-1)^N R_N \end{bmatrix} = \begin{bmatrix} -q_0 & p_0 \\ q_1 & -p_1 \\ \vdots & \vdots \\ -(-1)^N q_N & (-1)^N p_N \end{bmatrix},$$

using an analogous argument it can be shown that the second

column of the product $L \begin{bmatrix} -R_0 \\ R_1 \\ \vdots \\ -(-1)^N R_N \end{bmatrix}$ equals zero. In order to

prove that the first column of the product equals $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$, use $q_i =$

$$\sum_{j=0}^{N-i} p_{i+j}h_j, \quad i = 1, \dots, N. \quad \square$$

$$\tilde{\Pi} = \begin{bmatrix} -q_1 & p_1 & -q_2 & p_2 & \dots & \dots & -q_N & p_N & 0 & \dots \\ q_2 & -p_2 & \dots & \dots & q_N & -p_N & 0 & 0 & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \dots & \dots \\ (-1)^N q_N & -(-1)^N p_N & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Box I.

Example 4.2. We consider the system with $q(\xi) = \xi^2 + \xi + 1$, $p(\xi) = \xi^3 + 3\xi + 2$. In this case

Then $\tilde{\Pi} = \tilde{Y}^T \tilde{X}$. Moreover, there exists a left inverse L of the matrix consisting of the first N columns and $N + 1$ rows of \tilde{Y}^T , such that

$$\Pi(\zeta, \eta) = \begin{bmatrix} 1 & \zeta & \zeta^2 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}.$$

To construct a realization as in Proposition 4.1, we define

$$-L \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} = \begin{bmatrix} -\frac{p_{N-1}}{p_N} & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ -\frac{p_0}{p_N} & 0 & \dots & 0 \end{bmatrix}, \quad (15)$$

$$\tilde{Y} = \begin{bmatrix} -3 & 0 & -1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \end{bmatrix}$$

$$\tilde{X} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & \dots \end{bmatrix}.$$

Following the notation introduced in the proof of the proposition,

$$-L \begin{bmatrix} R_0 \\ -R_1 \\ \vdots \\ (-1)^N R_N \end{bmatrix} = \begin{bmatrix} q_{N-1} - p_{N-1} \frac{q_N}{p_N} & 0 \\ \vdots & \vdots \\ q_1 - p_1 \frac{q_N}{p_N} & 0 \\ q_0 - p_0 \frac{q_N}{p_N} & 0 \end{bmatrix}, \quad (16)$$

$$T := \begin{bmatrix} -3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad v := \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix},$$

which yields

$$L = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ -1 & 0 & 3 & -2 \end{bmatrix}.$$

The formulas (13) yield

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

$$C = [1 \ 0 \ 0], \quad D = 0. \quad \square$$

We now show how the observer canonical form arises from factorizations of $\tilde{\Pi}$.

Proposition 4.3. Let $q = 2$, and R be defined by (12). Define \tilde{Y} and \tilde{X} by

$$\tilde{Y} := \begin{bmatrix} 0 & 0 & \dots & 0 & (-1)^N & 0 & \dots \\ 0 & 0 & \dots & (-1)^{N-1} & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \dots \\ 0 & 1 & 0 & \ddots & \dots & \vdots & \dots \\ -1 & 0 & 0 & \dots & \dots & 0 & \dots \end{bmatrix}$$

and

$$\tilde{X} := \begin{bmatrix} q_N & -p_N & 0 & 0 & \dots & \dots & 0 & 0 & \dots & \dots \\ \vdots & \vdots & q_N & -p_N & \dots & \dots & \vdots & \vdots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \dots \\ q_1 & -p_1 & q_2 & -p_2 & \dots & \dots & q_N & -p_N & \dots & \dots \end{bmatrix}.$$

and

$$-Y_{N-1}^T = [(-1)^N \ 0_{1 \times (N-1)}] \quad (17)$$

$$(-1)^N R_N = (-1)^N [q_N \ -p_N].$$

Proof. The first claim is straightforward. To prove the second claim, define

$$L := \begin{bmatrix} 0 & \dots & 0 & (-1)^N & (-1)^N \frac{p_{N-1}}{p_N} \\ 0 & \dots & (-1)^{N-1} & 0 & (-1)^N \frac{p_{N-2}}{p_N} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & \dots & 0 & (-1)^N \frac{p_0}{p_N} \end{bmatrix}.$$

It is a matter of straightforward verification to check that the

columns of $-L \begin{bmatrix} R_0 \\ -R_1 \\ \vdots \\ (-1)^N R_N \end{bmatrix}$ are respectively $\begin{bmatrix} q_{N-1} - p_{N-1} \frac{q_N}{p_N} \\ \vdots \\ q_0 - p_0 \frac{q_N}{p_N} \end{bmatrix}$ and the zero vector. The claim is proved. \square

Example 4.4. We consider the same system of Example 4.2. The factorization of $\tilde{\Pi}$ is given by

$$\tilde{Y} = \begin{bmatrix} 0 & 0 & -1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \end{bmatrix}$$

$$\tilde{X} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & \dots \\ 1 & -3 & 1 & 0 & 0 & -1 & 0 & \dots \end{bmatrix}.$$

The matrix L is defined by

$$L := \begin{bmatrix} 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The formulas (15)–(17) yield

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 0. \quad \square$$

In order to show how the canonical controller- and controllability form arise in the framework developed in [3], we need to introduce some concepts and notation.

Considering the relationship between state- and behavioral controllability (see [7]), in the rest of this section we assume that the polynomials p and q corresponding to the kernel representation (12) are coprime. Under this assumption, it is well-known (see for example Lemma 2.4.10 of [5]) that there exist unique polynomials a and b such that $\deg(a) < \deg(p)$, $\deg(b) < \deg(q)$ and the Bézout equation $a(\xi)q(\xi) + b(\xi)p(\xi) = 1$ holds. Define now the two-variable polynomial

$$B(\zeta, \eta) := \frac{q(-\zeta)p(\eta) - p(-\zeta)q(\eta)}{\zeta + \eta}. \quad (18)$$

It is a matter of straightforward verification to check that the coefficient matrix B of (18) and that of the classical Bézoutian of p and q (see Section 8.4 of [8] for a definition), denoted with \tilde{B} , are related by

$$\tilde{B} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & (-1)^{N-2} & 0 \\ 0 & \cdots & \cdots & 0 & (-1)^{N-1} \end{bmatrix} \tilde{B}' =: J\tilde{B}'. \quad (19)$$

From the coprimality of p and q it follows that \tilde{B}' and consequently \tilde{B} is nonsingular.

In the statement of our main result, we also need the polynomials

$$r_{b,i}(\xi) := \xi^i b(\xi) - n_{b,i}(\xi)q(\xi)$$

$$r_{a,i}(\xi) := \xi^i a(\xi) - n_{a,i}(\xi)p(\xi) \quad i = 0, \dots, N-1, \quad (20)$$

where $n_{b,i}$ and $n_{a,i}$ are the quotients of the Euclidean division of $\xi^i b(\xi)$, respectively $\xi^i a(\xi)$, by $q(\xi)$, respectively $p(\xi)$. Note that $r_{b,0} = b$ and $r_{a,0} = a$.

Proposition 4.5. *Let $q = 2$, and R be defined by (12). Define $B(\zeta, \eta)$ and the remainders by (18) and (20), respectively. Define \tilde{Y} and \tilde{X} by*

$$\tilde{Y} := \begin{bmatrix} \tilde{B}^T & 0_{N \times \infty} \end{bmatrix}$$

and

$$\tilde{X} := \begin{bmatrix} \tilde{r}_0 & 0_{1 \times \infty} \\ \vdots & \\ \tilde{r}_{N-1} & 0_{1 \times \infty} \end{bmatrix},$$

where \tilde{r}_i is the coefficient vector of $[r_{b,i}(\xi) \ r_{a,i}(\xi)]$, $i = 0, \dots, N-1$.

Then $\tilde{\Pi} = \tilde{Y}^T \tilde{X}$. Moreover, there exists a left inverse L of the matrix consisting of the first N columns and $N+1$ rows of \tilde{Y}^T , such that

$$-L \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -\frac{p_0}{p_N} & \cdots & \cdots & -\frac{p_{N-1}}{p_N} \end{bmatrix}, \quad (21)$$

$$-L \begin{bmatrix} R_0 \\ -R_1 \\ \vdots \\ (-1)^N R_N \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \frac{1}{p_N} & 0 \end{bmatrix} \quad (22)$$

and

$$-Y_{N-1}^T = [q_0 p_N - p_0 q_N \quad \cdots \quad q_{N-1} p_N - p_{N-1} q_N] \quad (23)$$

$$(-1)^N R_N = (-1)^N [q_N \quad -p_N],$$

i.e. the Eq. (11) are in canonical controller form.

Proof. To prove the first claim, note that $X(\eta) \begin{bmatrix} p(\eta) \\ q(\eta) \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ \eta^{N-1} \end{bmatrix}$.

Consequently, from the definition of Y it follows that

$$(\zeta + \eta)Y(\zeta)^T X(\eta) \begin{bmatrix} p(\eta) \\ q(\eta) \end{bmatrix}$$

$$= (\zeta + \eta)Y(\zeta)^T \begin{bmatrix} 1 \\ \vdots \\ \eta^{N-1} \end{bmatrix} = (\zeta + \eta)B(\zeta, \eta)$$

$$= p(\eta)q(-\zeta) - q(\eta)p(-\zeta).$$

Multiplying $\Pi(\zeta, \eta)$ on the right by $\begin{bmatrix} p(\eta) \\ q(\eta) \end{bmatrix}$ we also obtain $p(\eta)q(-\zeta) - q(\eta)p(-\zeta)$. This argument proves that

$$(\zeta + \eta)\Pi(\zeta, \eta) \begin{bmatrix} p(\eta) \\ q(\eta) \end{bmatrix} = (\zeta + \eta)Y(\zeta)^T X(\eta) \begin{bmatrix} p(\eta) \\ q(\eta) \end{bmatrix},$$

and consequently

$$\Pi(\zeta, \eta) \begin{bmatrix} p(\eta) \\ q(\eta) \end{bmatrix} = Y(\zeta)^T X(\eta) \begin{bmatrix} p(\eta) \\ q(\eta) \end{bmatrix}. \quad (24)$$

Define $\Theta(\zeta, \eta) := Y(\zeta)^T X(\eta)$; then (24) is equivalent with the BDFs L_Θ and L_Π being equivalent along $(\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}), \mathcal{B})$, that is, $L_\Theta(\varphi, w) = L_\Pi(\varphi, w)$ for all $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and all $w \in \mathcal{B}$.

Now let $F(\zeta)^T G(\eta)$ be any minimal factorization of $\Pi(\zeta, \eta)$; we now prove that since L_Θ and L_Π are equivalent along $(\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}), \mathcal{B})$, there exists a nonsingular $T \in \mathbb{R}^{N \times N}$ such that $TG(\frac{d}{dt})w = X(\frac{d}{dt})w$ for all $w \in \mathcal{B}$. In order to do this, observe that for all $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and all $w \in \mathcal{B}$ it holds that

$$\left(X \left(\frac{d}{dt} \right) w \right) (0)$$

$$= 0 \implies \left(F \left(\frac{d}{dt} \right) \varphi \right) (0)^T \left(G \left(\frac{d}{dt} \right) w \right) (0) = 0,$$

and consequently because of the arbitrariness of φ , it holds that for all $w \in \mathcal{B}$

$$\left(X \left(\frac{d}{dt} \right) w \right) (0) = 0 \implies \left(G \left(\frac{d}{dt} \right) w \right) (0) = 0.$$

This implies (see Lemma B.2 p. 1081 of [1]) that there exist $T \in \mathbb{R}^{n \times n}$ and $K \in \mathbb{R}^{n \times 1}[\xi]$ such that $G(\xi) = TX(\xi) + K(\xi)R(\xi)$. Since G arises from a minimal factorization of $\Pi(\zeta, \eta)$, the largest power of ξ in $G(\xi)$ is $N-1$, and consequently $K(\xi) = 0$. This proves that $X(\xi) = TG(\xi)$, and consequently that $X(\frac{d}{dt}) = TG(\frac{d}{dt})$ as polynomial differential operators on $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$.

Now observe that $F(\zeta)^T T^{-1}TG(\eta) = F(\zeta)^T T^{-1}X(\eta)$, from which the equality $F(\zeta)^T T^{-1} = Y(\zeta)^T$ follows in a straightforward way. This concludes the proof of the first part of the claim.

In order to prove the rest of the statements, we proceed as follows. It is a matter of straightforward verification to

check that the last row of the coefficient matrix of $B(\zeta, \eta)$ is $[p_N q_0 - p_0 q_N \quad \dots \quad p_N q_{N-1} - p_{N-1} q_N]$; this proves the claim on Y_{N-1}^T .

From J and \tilde{B}' defined by (19), define

$$v := \frac{1}{(-1)^N p_N} \tilde{B}'^{-1} J \begin{bmatrix} -p_0 \\ \vdots \\ -(-1)^{N-1} p_{N-1} \end{bmatrix},$$

and

$$L := [\tilde{B}'^{-1} J \quad v].$$

It is a matter of straightforward verification to check that L is a left inverse of $[Y_0 \quad \dots \quad Y_{N-1} \quad 0_{n \times p}]^T$; we now show that it satisfies the remaining equalities in (13). It is straightforward to verify that

$$L \begin{bmatrix} -p_0 \\ \vdots \\ -(-1)^N p_N \end{bmatrix} = \tilde{B}'^{-1} J \begin{bmatrix} -p_0 \\ \vdots \\ -(-1)^{N-1} p_{N-1} \end{bmatrix} - (-1)^N p_N v = 0.$$

Now note that

$$L \begin{bmatrix} q_0 \\ \vdots \\ (-1)^N q_N \end{bmatrix} = \tilde{B}'^{-1} J \begin{bmatrix} q_0 \\ \vdots \\ (-1)^{N-1} q_{N-1} \end{bmatrix} + (-1)^N q_N v = 0$$

if and only if

$$\begin{bmatrix} q_0 \\ \vdots \\ (-1)^{N-1} q_{N-1} \end{bmatrix} + \frac{q_N}{p_N} \begin{bmatrix} -p_0 \\ \vdots \\ -(-1)^{N-1} p_{N-1} \end{bmatrix} = J \tilde{B}' \begin{bmatrix} 0 \\ \vdots \\ \frac{1}{p_N} \end{bmatrix}.$$

To prove this last equality, observe that the last column of the coefficient matrix of $B(\zeta, \eta)$ is

$$[p_N q_0 - p_0 q_N \quad \dots \quad (-1)^{N-1} (p_N q_{N-1} - p_{N-1} q_N)]^T.$$

$$\text{This argument proves that } -L \begin{bmatrix} R_0 \\ -R_1 \\ \vdots \\ (-1)^N R_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{p_N} \\ 0 \end{bmatrix}.$$

To prove the claim on the state matrix of the realization, write

$$\begin{aligned} -L \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} &= -[\tilde{B}'^{-1} J \quad v] \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} \\ &= -\tilde{B}'^{-1} J \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-2}^T \end{bmatrix} + v Y_{N-1}^T. \end{aligned}$$

We now prove that the last expression equals A_p , the row-companion matrix associated with the polynomial p . This is true if and only if

$$J \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-2}^T \end{bmatrix} + \tilde{B}' v Y_{N-1}^T = \tilde{B}' A_p.$$

From the definition of v and some straightforward manipulations it can be verified that the left-hand side equals $A_p^T \tilde{B}'$. Consequently, the equality holds if and only if

$$A_p^T \tilde{B}' = \tilde{B}' A_p;$$

this however is a well-known result, see Corollary 4.4 p. 190 of [9]. This concludes the proof. \square

Example 4.6. We consider the same system of Example 4.2. The two-variable polynomial $B(\zeta, \eta)$ has coefficient matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix};$$

Consequently,

$$\tilde{Y}^T = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It can be verified that the Bézout equation $a(\xi)q(\xi) + b(\xi)p(\xi) = 1$ has solutions $a(\xi) = 1 - \frac{1}{3}\xi + \frac{1}{3}\xi^2$ and $b(\xi) = -\frac{1}{3}\xi$. This leads to the matrix

$$X(\xi) := \begin{bmatrix} -\frac{1}{3}\xi & 1 - \frac{1}{3}\xi + \frac{1}{3}\xi^2 \\ \frac{1}{3} + \frac{1}{3}\xi & -\frac{2}{3} - \frac{1}{3}\xi^2 \\ -\frac{1}{3} & \frac{2}{3} + \frac{1}{3}\xi \end{bmatrix},$$

which yields

$$\tilde{X} = \begin{bmatrix} 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \end{bmatrix}.$$

The left-inverse matrix L equals

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}.$$

It can be verified that the realization defined by (21)–(23) is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ C = [1 \quad 1 \quad 1], \quad D = 0. \quad \square$$

Finally, we examine the canonical controllability form.

Proposition 4.7. Let $q = 2$, and R be defined by (12). Define $B(\zeta, \eta)$ and the remainders by (18) and (20), respectively. Let

$$P := \begin{bmatrix} p_N & 0 & \dots & 0 \\ p_{N-1} & p_N & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ p_1 & \dots & \dots & p_N \end{bmatrix},$$

and define \tilde{Y} and \tilde{X} by

$$\tilde{Y} := [P^{-T} \tilde{B}^T \quad 0_{N \times \infty}]$$

and

$$\tilde{X} := [P \tilde{X}' \quad 0_{N \times \infty}],$$

where $\tilde{X}' := \begin{bmatrix} \tilde{r}_0 \\ \vdots \\ \tilde{r}_{N-1} \end{bmatrix}$, with \tilde{r}_i being the coefficient vector of

$[r_{b,i}(\xi) \quad r_{a,i}(\xi)]$, $i = 0, \dots, N-1$. Then $\tilde{T} = \tilde{Y}^T \tilde{X}$.

Moreover, there exists a left inverse L of the matrix consisting of the first N columns and $N+1$ rows of \tilde{Y}^T , such that

$$-L \begin{bmatrix} 0 \\ Y_0^T \\ \vdots \\ Y_{N-1}^T \end{bmatrix} = \begin{bmatrix} -\frac{p_{N-1}}{p_N} & 1 & \dots & 0 \\ \frac{p_N}{p_N} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -\frac{p_1}{p_N} & \dots & 0 & 1 \\ \frac{p_N}{p_N} & \dots & \dots & 0 \\ -\frac{p_0}{p_N} & \dots & \dots & 0 \end{bmatrix},$$

$$-L \begin{bmatrix} R_0 \\ -R_1 \\ \vdots \\ (-1)^N R_N \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \\ \frac{1}{p_N} & 0 \end{bmatrix},$$

and

$$-Y_{N-1}^T = [q_0 p_N - p_0 q_N \quad \dots \quad q_{N-1} p_N - p_{N-1} q_N]$$

$$(-1)^N R_N = (-1)^N [q_N \quad -p_N],$$

i.e. the Eq. (11) are in canonical controllability form.

Proof. The first claim follows in a straightforward manner from the factorization proved in Proposition 4.5. The other claims can

be proven in two ways. One is to define

$$L := [P \tilde{B}^{-1} \quad v] \quad \text{and}$$

$$v := \frac{1}{(-1)^N p_N} P \tilde{B}^{-1} \begin{bmatrix} -p_0 \\ \vdots \\ -(-1)^{N-1} p_{N-1} \end{bmatrix},$$

and proceed analogously to the argument used in proving Proposition 4.5. The second way to prove the claim is to use the well-known relations between the controller and the controllability form (see Section 2.4.1 of [5]). Indeed, denoting by x_c the state variable for the controller realization of Proposition 4.5, the state variable of the controllability realization is obtained precisely by Px_c , see Fig. 2.4-3 p. 129 of [5]. \square

5. Conclusions

We have shown how the classical canonical (controllability/controller and observability/observer) realizations fit in the framework for the computation of state maps and state equations initiated in [3].

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